GCE

Further Mathematics A

Y535/01: Additional Pure Mathematics

Advanced Subsidiary GCE

Mark Scheme for June 2019

This mark scheme is published as an aid to teachers and students, to indicate the requirements of the examination. It shows the basis on which marks were awarded by examiners. It does not indicate the details of the discussions which took place at an examiners' meeting before marking commenced.

All examiners are instructed that alternative correct answers and unexpected approaches in candidates' scripts must be given marks that fairly reflect the relevant knowledge and skills demonstrated.

Mark schemes should be read in conjunction with the published question papers and the report on the examination.

Annotations and abbreviations

Annotation in scoris	**Meaning**
✓and ✗	
BOD	Benefit of doubt
FT	Follow through
ISW	Ignore subsequent working
M0, M1	Method mark awarded 0, 1
A0, A1	Accuracy mark awarded 0, 1
B0, B1	Independent mark awarded 0, 1
SC	Special case
^	Omission sign
MR	Misread
Highlighting	
Other abbreviations in mark scheme	**Meaning**
E1	Mark for explaining a result or establishing a given result
dep*	Mark dependent on a previous mark, indicated by *
cao	Correct answer only
oe	Or equivalent
rot	Rounded or truncated
soi	Seen or implied
www	Without wrong working
AG	Answer given
awrt	Anything which rounds to
BC	By Calculator
DR	This question included the instruction: In this question you must show detailed reasoning.

Subject-specific Marking Instructions for AS Level Further Mathematics A

a Annotations should be used whenever appropriate during your marking. The A, M and B annotations must be used on your standardisation scripts for responses that are not awarded either 0 or full marks. It is vital that you annotate standardisation scripts fully to show how the marks have been awarded. For subsequent marking you must make it clear how you have arrived at the mark you have awarded.

b An element of professional judgement is required in the marking of any written paper. Remember that the mark scheme is designed to assist in marking incorrect solutions. Correct solutions leading to correct answers are awarded full marks but work must not be judged on the answer alone, and answers that are given in the question, especially, must be validly obtained; key steps in the working must always be looked at and anything unfamiliar must be investigated thoroughly. Correct but unfamiliar or unexpected methods are often signalled by a correct result following an apparently incorrect method. Such work must be carefully assessed. When a candidate adopts a method which does not correspond to the mark scheme, escalate the question to your Team Leader who will decide on a course of action with the Principal Examiner.
If you are in any doubt whatsoever you should contact your Team Leader.

c The following types of marks are available.

M
A suitable method has been selected and *applied* in a manner which shows that the method is essentially understood. Method marks are not usually lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. In some cases the nature of the errors allowed for the award of an M mark may be specified.

A
Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated Method mark is earned (or implied). Therefore M0 A1 cannot ever be awarded.

B
Mark for a correct result or statement independent of Method marks.

E
Mark for explaining a result or establishing a given result. This usually requires more working or explanation than the establishment of an unknown result.

Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored. Sometimes this is reinforced in the mark scheme by the abbreviation isw. However, this would not apply to a case where a candidate passes through the correct answer as part of a wrong argument.

d When a part of a question has two or more ‘method’ steps, the M marks are in principle independent unless the scheme specifically says otherwise; and similarly where there are several B marks allocated. (The notation ‘dep*’ is used to indicate that a particular mark is dependent on an earlier, asterisked, mark in the scheme.) Of course, in practice it may happen that when a candidate has once gone wrong in a part of a question, the work from there on is worthless so that no more marks can sensibly be given. On the other hand, when two or more steps are successfully run together by the candidate, the earlier marks are implied and full credit must be given.

e The abbreviation FT implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A and B marks are given for correct work only – differences in notation are of course permitted. A (accuracy) marks are not given for answers obtained from incorrect working. When A or B marks are awarded for work at an intermediate stage of a solution, there may be various alternatives that are equally acceptable. In such cases, what is acceptable will be detailed in the mark scheme. If this is not the case please, escalate the question to your Team Leader who will decide on a course of action with the Principal Examiner.
Sometimes the answer to one part of a question is used in a later part of the same question. In this case, A marks will often be 'follow through'. In such cases you must ensure that you refer back to the answer of the previous part question even if this is not shown within the image zone. You may find it easier to mark follow through questions candidate-by-candidate rather than question-by-question.

f Unless units are specifically requested, there is no penalty for wrong or missing units as long as the answer is numerically correct and expressed either in SI or in the units of the question. (e.g. lengths will be assumed to be in metres unless in a particular question all the lengths are in km, when this would be assumed to be the unspecified unit.) We are usually quite flexible about the accuracy to which the final answer is expressed; over-specification is usually only penalised where the scheme explicitly says so. When a value is given in the paper only accept an answer correct to at least as many significant figures as the given value. This rule should be applied to each case. When a value is not given in the paper accept any answer that agrees with the correct value to 2 s.f. Follow through should be used so that only one mark is lost for each distinct accuracy error, except for errors due to premature approximation which should be penalised only once in the examination. There is no penalty for using a wrong value for g. E marks will be lost except when results agree to the accuracy required in the question.

g Rules for replaced work: if a candidate attempts a question more than once, and indicates which attempt he/she wishes to be marked, then examiners should do as the candidate requests; if there are two or more attempts at a question which have not been crossed out, examiners should mark what appears to be the last (complete) attempt and ignore the others. NB Follow these maths-specific instructions rather than those in the assessor handbook.

h For a genuine misreading (of numbers or symbols) which is such that the object and the difficulty of the question remain unaltered, mark according to the scheme but following through from the candidate's data. A penalty is then applied; 1 mark is generally appropriate, though this may differ for some units. This is achieved by withholding one A mark in the question. Marks designated as cao may be awarded as long as there are no other errors. E marks are lost unless, by chance, the given results are established by equivalent working. 'Fresh starts' will not affect an earlier decision about a misread. Note that a miscopy of the candidate's own working is not a misread but an accuracy error.

i If a calculator is used, some answers may be obtained with little or no working visible. Allow full marks for correct answers (provided, of course, that there is nothing in the wording of the question specifying that analytical methods are required). Where an answer is wrong but there is some evidence of method, allow appropriate method marks. Wrong answers with no supporting method score zero. If in doubt, consult your Team Leader.

j If in any case the scheme operates with considerable unfairness consult your Team Leader.

Question			Answer	Marks	AOs	Guidance
1	(a)		$N = 11\ 101\ 110\ 011\ 100_2$	B1 [1]	1.1	BC
	(b)		$7_{10} = 111_2$ $11\ 101\ 110\ 011\ 100_2 = 100010000100_2 \times 111_2$ $\Rightarrow 7 \mid N$	M1 A1	1.1 2.1	**soi** any recognition N is made up of blocks of 111 Result and conclusion (**FT** blocks of 111)
			Alternative method			
			$N = (2^2 + 2^3 + 2^4) + (2^7 + 2^8 + 2^9) + (2^{11} + 2^{12} + 2^{13})$ and working mod 7	M1		
			$\equiv (4 + 1 + 2) + (2 + 4 + 1) + (4 + 1 + 2) \pmod 7 \equiv 0$	A1		
				[2]		
2	(a)		Limit = 4	B1 [1]	1.1	BC
	(b)		Setting $b_{n+1} = b_n = 9$ throughout Solving $9 = \sqrt{9} + \dfrac{k}{\sqrt{9}}$ $k = 18$	M1 M1 A1	3.1a 1.1a 1.1	 **soi** **BC** or by inspection
			Alternative method			
			Search method for $b_{n+1} = \sqrt{b_n} + \dfrac{k}{\sqrt{b_n}}$ with various k's	M1		
			Evidence of systematic approach (e.g. $k = 9 \to 6.11\ldots$, $k = 20 \to 9.56\ldots$) $k = 18$	M1 A1		
				[3]		
3	(a)		**x** and **y** are parallel $\mathbf{x} \times \mathbf{y} = xy \sin\theta\, \mathbf{u}$ (where **u** is a unit vector) $= 0 \Rightarrow$ (since $x, y \neq 0$) $\sin\theta = 0 \Rightarrow \theta = 0$ (or π) and $\mathbf{x} \parallel \mathbf{y}$	B1 E1 [2]	1.2 2.4	
	(b)		$\mathbf{r} = \mathbf{a} + t\,\mathbf{d} \Rightarrow \mathbf{r} - \mathbf{a} = t\,\mathbf{d} \Leftrightarrow (\mathbf{r} - \mathbf{a}) \parallel \mathbf{d}$ Then, by **(a)**, $(\mathbf{r} - \mathbf{a}) \times \mathbf{d} = \mathbf{0}$	M1 A1	2.1 2.2a	(Since one vector a multiple of the other) No statement required that neither vector is zero Condone lack of ⇔-ness to explanation
			Alternative method			
			$\mathbf{r} = \mathbf{a} + t\,\mathbf{d} \Rightarrow \mathbf{r} - \mathbf{a} = t\,\mathbf{d}$ and $\times\,\mathbf{d}$ both sides Conclusion follows from $\mathbf{d} \times \mathbf{d} = \mathbf{0}$	M1 A1		
				[2]		

Question			Answer	Marks	AOs	Guidance
4			$u_{n+1} - 2u_n = n^2$			
			Complementary Soln. is $u_n = A \times 2^n$	B1	1.1	
			For Particular Soln., try $u_n = an^2 + bn + c$	M1	2.1	
			Substg. into given r.r. for both u_{n+1} and u_n	M1	1.1	
			$an^2 + 2an + a + bn + b + c - 2(an^2 + bn + c) = n^2$			
			Comparing coeffts.	M1	1.1	
			$a = -1$, $b = -2$, $c = -3$ so that PS is $u_n = -(n^2 + 2n + 3)$	A1	2.2a	
			General Soln. is $u_n = A \times 2^n - (n^2 + 2n + 3)$	B1	1.1	**FT**
			Use of initial term to evaluate A	M1	1.1a	
			$u_1 = 1 = 2A - (1 + 2 + 3) \Rightarrow A = \frac{7}{2}$			
			and Soln. is $u_n = 7 \times 2^{n-1} - (n^2 + 2n + 3)$ **oe**	A1	1.1	**cao** If all correct, the final A mark may be awarded at the previous line
			Alternative method			
			$u_n = A \times 2^n + an^2 + bn + c$	B1 M1		CS, PS
			$\{u_n\} = \{1, 3, 10, 29, 74, \ldots\}$	M1		Using correct first four (or five?) terms in system of equations
			Setting up system of equations: $1 = 2A + a + b + c$; $3 = 4A + 4a + 2b + c$; $10 = 8A + 9a + 3b + c$; $29 = 16A + 16a + 4b + c$	M1 M1		**M1** for at least two; **M2** for all four (or five?)
			Solving system of equations	M1		**BC**
			$A = \frac{7}{2}$ and $a = -1$, $b = -2$, $c = -3$	A1 A1		
				[8]		

Question			Answer	Marks	AOs	Guidance
5			Area $\Delta OAB = \frac{1}{2} \mid \mathbf{a} \times \mathbf{b} \mid$	M1	3.1a	Use of vector product **oe** for one simple Δ area
			where $\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}$	B1	1.1	A correct, relevant vector product calculation
			$= \frac{1}{2}\sqrt{17}$	A1	1.1	First correct simple Δ area (exact answer justified)
			Area ΔOAC = Area $\Delta OBC = \frac{1}{2}\sqrt{17}$ similarly	B1	1.1	$\mathbf{a} \times \mathbf{c} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ -2 \end{pmatrix}$, $\mathbf{b} \times \mathbf{c} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix}$
			Area $\Delta ABC = \frac{1}{2} \mid (\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a}) \mid$ e.g.	M1	1.1a	Final, complicated Δ area attempted
			where $(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a}) = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$			$\mathbf{c} - \mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$
			$= \frac{1}{2}\sqrt{3}$	A1	1.1	(First **B1** can be earned for this area if not otherwise) Must follow from correct vector product
			Surface area of T is then $3 \times \frac{1}{2}\sqrt{17} + \frac{1}{2}\sqrt{3}$	M1	3.1a	Sum of four calculated Δ areas
			$= \frac{1}{2}\sqrt{3}\left(\sqrt{3}\sqrt{17}+1\right) = \frac{1}{2}\sqrt{3}\left(\sqrt{51}+1\right)$	A1	1.1	**AG** fully legitimately obtained
				[8]		
			Note that *OAB*, *OAC*, *OBC* are congruent isosceles Δs with sides 3, 3, $\sqrt{2}$, while *ABC* is an equilateral Δ of side $\sqrt{2}$			

Question			Answer	Marks	AOs	Guidance
6	(a)		**DR** (Working mod 101 throughout) $16x \equiv 5 \equiv 106 \ldots$	**M1**	1.1	Adding multiples of 101 at any stage
			$\equiv 1520$	**A1**	2.1	Finding a multiple of 16
			Explanation that we can divide by 16 since hcf(16, 101) = 1	**E1**	2.4	Explained appropriately at any stage
			$\Rightarrow\ x \equiv 95 \pmod{101}$ or $x = 101n + 95$ or any other valid form	**A1**	2.2a	(Use of hcf(16, 101) = 1 to justify other attributes **E0**)
			Alternative method I Done in stages			
			e.g. $16x \equiv 5 \equiv 106 \ldots \Rightarrow 8x \equiv 53$	**M1**		
			$\Rightarrow 8x \equiv 154 \Rightarrow 4x \equiv 77 \equiv 178$	**A1**		Must be evidence of repeated divisions (correct ≥ twice)
			$\Rightarrow 2x \equiv 89 \equiv 190 \Rightarrow x \equiv 95 \pmod{101}$ etc.	**A1**		
			Explanation that we can divide by 2 since hcf(2, 101) = 1	**E1**		Explained appropriately at any stage (once will suffice)
			Alternative method II Using reciprocal/inverse			
			Finding $16^{-1} \pmod{101} = 19$	**M1 A1**		
			Multiplying throughout $16x \equiv 5 \pmod{101}$ by 19 $\Rightarrow x \equiv 95 \pmod{101}$	**M1 A1**		
				[4]		
	(b)	(i)	**DR** $95x \equiv 6 \Rightarrow -6x \equiv 6$	**M1**	3.1a	
			$\Rightarrow x \equiv -1 \pmod{101}$ **oe**	**A1**	1.1	
			Alternative method I Using **(a)**			
			Multg. throughout by 16 $\Rightarrow 5x \equiv 96 \pmod{101}$			**Or** by noting that this is $5x \equiv -5 \pmod{101}$
			Multg. throughout by 81 $\Rightarrow 405x \equiv x \equiv 100 \pmod{101}$	**M1 A1**		Complete method **NB** $81 \times 5 = 405 \equiv 1 \pmod{101}$
			Alternative method II Using reciprocal/inverse			
			Finding $95^{-1} \pmod{101} = 84$ Multg. throughout by 84	**M1**		Complete method
			$\Rightarrow x \equiv 100 \pmod{101}$	**A1**		**NB** $84 \equiv 16 \times 81 \pmod{101}$
				[2]		
	(b)	(ii)	Using part **(a)**'s answer, $95 \times 16 \equiv 5 \pmod{101}$	**M1**	2.2a	Mark may be earned by solving the linear congruence from scratch; must be a *complete* method
			$\Rightarrow x \equiv 16 \pmod{101}$	**A1**	1.1	
				[2]		

<table>
<tr><th colspan="3">Question</th><th>Answer</th><th>Marks</th><th>AOs</th><th>Guidance</th></tr>
<tr><td>7</td><td>(a)</td><td></td><td><table>
<tr><th></th><th>1</th><th>5</th><th>7</th><th>11</th><th>13</th><th>17</th></tr>
<tr><th>1</th><td>1</td><td>5</td><td>7</td><td>11</td><td>13</td><td>17</td></tr>
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<tr><th>13</th><td>13</td><td>11</td><td>1</td><td>17</td><td>7</td><td>5</td></tr>
<tr><th>17</th><td>17</td><td>13</td><td>11</td><td>7</td><td>5</td><td>1</td></tr>
</table></td><td>M1

A1

A1

A1
[4]</td><td>1.1a

1.1

1.1

1.1</td><td>General form/layout

At least R_1 and C_1 correct (shaded)

LSP observed

All correct</td></tr>
<tr><td></td><td>(b)</td><td></td><td>Closure ✓ since there are no new elements in the table
[Associativity assumed]
Identity is 1

Inverses: (5, 11) and (7, 13) are inverse-pairs; 17 is self-inverse

(⇒ Group)</td><td>B1

B1

B1

[3]</td><td>2.2a

1.1

2.4</td><td>Statements like “Closure from the table” are insufficient

Statements such as “1 appears in every row and column” are insufficient since one should justify that each element’s left-inverse and right-inverse are the same.</td></tr>
<tr><td></td><td>(c)</td><td></td><td>Elements: 1 5 7 11 13 17
Orders: 1 6 3 6 3 2</td><td>M1
A1

[2]</td><td>1.1a
1.1</td><td>At least 2 (non-identity) elements correct
All 5 (non-identity) elements correct
Ignore missing (or incorrect) order for 1</td></tr>
<tr><td></td><td>(d)</td><td></td><td>It has (at least) one element of order 6
or noting that there is a generator (5 or 11)</td><td>B1
[1]</td><td>2.4</td><td></td></tr>
<tr><td></td><td>(e)</td><td>(i)</td><td>e.g. S_3, the group of six permutations of 3 symbols
or D_3, the (dihedral) group of symmetries of the triangle
or the product group $\mathbb{Z}_3 \otimes \mathbb{Z}_2$</td><td>B1
[1]</td><td>1.2</td><td>or the corresponding transformations

Allow wordy descriptions if complete</td></tr>
<tr><td></td><td></td><td>(ii)</td><td>The non-cyclic group has elements of orders (1), 2, 2, 2, 3, 3
or noting that all (non-identity) elements have order 2 or 3
or this group is not abelian (or non-commutative) (⇒ it is non-cyclic)</td><td>B1
[1]</td><td>2.4</td><td></td></tr>
</table>

Question			Answer	Marks	AOs	Guidance
8	(a)		$\overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = \begin{pmatrix} 8q-4p+3 \\ q-p+5 \\ 4q-3p+4 \end{pmatrix}$	M1	3.3	± Attempted **soi**
			$z = (PQ)^2 = (8q - 4p + 3)^2 + (q - p + 5)^2 + (4q - 3p + 4)^2$ $= (64q^2 + 16p^2 + 9 - 64pq - 24p + 48q)$ $+ (q^2 + p^2 + 25 - 2pq - 10p + 10q)$ $+ (16q^2 + 9p^2 + 16 - 24pq - 24p + 32q)$	M1	1.2	Clear attempt to square at least two three-term brackets
			$= 81q^2 + 26p^2 + 50 - 90pq - 58p + 90q$	A1 [3]	1.1	**AG** from fully supported (visible) working
	(b)		$\dfrac{\partial z}{\partial p} = 52p - 90q - 58$ $\quad \dfrac{\partial z}{\partial q} = 162q - 90p + 90$	B1 B1	1.1 1.1	
			Setting both p.d.s to zero and solving $\begin{matrix} 26p - 45q = 29 \\ 45p - 81q = 45 \end{matrix}$ simultaneously	M1	3.1a	**BC**
			$p = 4, \ q = \frac{5}{3}$	A1 [4]	1.1	
	(c)		(Diagram may consist of two skew lines; P on one, Q on the other.) Moving P, Q in "opposite" directions along their lines gives z indefinitely large, hence stationary point is not a maximum	E1	3.4	**Or** z-p-q (∪-shaped) paraboloid drawn **Or** z-p AND z-q (∪-shaped) parabola drawn
			Symmetric properties of P, Q (i.e. p, q) gives both max or both min so not a saddle-point	E1	3.4	
			Alternative method I z-p-q (∪-shaped) paraboloid **OR** z-p AND z-q (∪-shaped) parabola drawn Noting surface has a minimum for each section (≥ 2 shown)	E1 E1		
			Alternative method II Skew lines have a minimum distance, so z must have a minimum There is only one stationary point in **(b)**, so it must be this minimum and not either a max. or a saddle point.	E1 E1		
				[2]		
	(d)		Substg. back $p = 4, \ q = \frac{5}{3}$ into expression for z	M1	1.1a	
			$\Rightarrow z = 9$ and Sh. Dist. = 3 (m)	A1 [2]	2.2a	**cao**

Question			Answer	Marks	AOs	Guidance
	(e)		e.g. Because they are modelled as spheres, for any value of p and q the distance between them will simply be less than in the original model.	**M1**	**3.5c**	
			the shortest distance is now $3 - 1 = 2$ (m)	**A1** **[2]**	**1.1**	**ft (d)**'s answer **Or** statement distance is 1 m less

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